The form of cavities for mode (2.1) is shown in Fig.2 for the same σ but various β. The solid lines relate to steady cavities, the dash lines for $\beta=-0.1$, and the dash-dot lines for $\beta=-0.02$. The cavities at $\sigma=0.136$, 0.25, 0.5 are denoted by numerals *1*, *2*, and *3*.

In Fig.3 the dependence $L(\tau)$ for $A=0.023$; $\omega=0.063$ for mode (2.3) are shown by solid lines, and the dash lines relate to calculations of the steady state for the same $\sigma(\tau)$. The distribution of $r=\frac{U^2}{1+\sigma}$ along the cavity for mode (2.1) for $\beta=-0.02$ are given in Fig.4; curves *1* and *2* relate to $\tau=17$ and 23.

The convenience of judging the degree of unsteadiness of cavities using the value of r is obvious.

We may add that owing to the weak dependence of the form of the cavity on the shape of the cavitating body results of unsteady cavitational flows given here can be generalized considerably.

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THE STEADY SPECTRA OF PARTICLES IN DISPERSIBLE SYSTEMS WITH COAGULATION AND FRAGMENTATION*

V.N. PISKUNOV

The formation of a steady dimensional distribution of particles (particle spectra) is dispersible systems with coagulation and fragmentation is considered. The relation between versions of the kinetic equation that defines these processes is traced. An analytical solution is obtained for the parametric set of coagulation coefficients and the velocities of paired fragmentation. The steady spectrum of particles is investigated in the case when the fragmentation is of the multiple type.

The kinetic equation of coagulation with fragmentation in the case when the rate of particle supply to the system to compensate for the fragmented particles is linear with respect to their concentration was first formulated in /1/. The fragmentation process can stabilize a coagulating dispersed system, and result in the formation of steady spectra. Some analytical results on the behaviour of systems with coagulation and fragmentation were obtained in /2—5/.

1. The variation with time t of the particle spectrum in three-dimensionally homogeneous systems with coagulation and fragmentation is defined by the kinetic equation

$$\partial c(g,t)/\partial t=S(c;g,t)+Q(c;g,t) \quad (1.1)$$

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where $c(g,t)$ is the concentration of particles of mass g in unit volume (the particle spectrum); the operator S defines the input to the balance of the coagulation process, and the operator Q is the input of fragmentation. As a result of binary collisions the coagulation according to Smoluchowski's theory is given by (see, for example, /6/)

$$S(c;g,t)=\frac{1}{2}\int_0^g K(g-n,n)\,c(g-n)\,c(n)\,dn-c(g,t)\int_0^\infty K(g,n)\,c(n,t)\,dn \tag{1.2}$$

where $K(g,n)$ are the coagulation coefficients.

The operator Q can be expressed in two forms. The first of these is /1/

$$Q(c;g,t)=\int_g^\infty \gamma(g,n)\,c(n,t)\,dn-\frac{c(g,t)}{g}\int_0^g n\gamma(n,g)\,dn \tag{1.3}$$

The function $\gamma(n,g)$ defines the rate of input into the system of particles of mass n, produced by the fragmentation of particles of mass g. Obviously $\gamma(n,g)=0$ when $n>g$. It is convenient to represent the quantity $\gamma(n,g)$ in the form /6/

$$\gamma(n,g)=\frac{\theta(n,g)}{\tau(g)},\quad \int_0^g n\theta(n,g)\,dn=g \tag{1.4}$$

where $\tau(g)$ has the meaning of the lifetime of particle of mass g, and $\theta(n,g)$ defines the spectrum of particles formed by the fragmentation. The integral condition in (1.4) is a consequence of the combined mass in the process of fragmentation. The second form of the operator Q applies to the case of fragmentation into two particles. The discrete version of the kinetic equation is given in /2/, and for a continuous kinetic equation it is /5/

$$Q(c;g,t)=\int_g^\infty f(g,n-g)\,c(n,t)\,dn-\frac{c(g,t)}{2}\int_0^g f(g-n,n)\,dn \tag{1.5}$$

Here $f(g,n)$ defines the fragmentation rate of particle $(g+n)$ into g and n. By its meaning the function f must be symmetric relative to its arguments, i.e. $f(g,n)=f(n,g)$.

Formula (1.5) is a special case of (1.3). To prove this we transform the second term on the right side of (1.5), taking into account the symmetry of $f(g,n)$ and the equality $\gamma(n,g)=f(n,g-n)$, and obtain

$$\frac{1}{2}\int_0^g f(g-n,n)\,dn=\frac{1}{2g}\int_0^g[(g-n)f(g-n,n)+nf(g-n,n)]\,dn=\frac{1}{g}\int_0^g n\gamma(n,g)\,dn$$

Then, if

$$\gamma(n,g)=\gamma(g-n,g) \tag{1.6}$$

it is possible to use the operator Q in the form (1.5).

Condition (1.6) indicates that in fragmentation particles g form the same number of particles n and $(g-n)$ (in particular (1.6) is satisfied for fragmentation into two particles). The most general form of the operator Q is thus (1.3). Whenever $\gamma(n,g)$ has the property of symmetry (1.6), formulae (1.3) and (1.5) are equivalent.

Besides the rate of processes of coagulation $K(g,n)$ and fragmentation $\gamma(n,g)$ one of the basic quantities that determine the solution of this problem is the mass of the particles per unit volume.

$$\rho=\int_0^\infty gc(g,t)\,dg \tag{1.7}$$

The substitution of explicit expressions for the operators S and Q into (1.1) shows that the above does not change with time. The steady state spectra of $c^s(g)$ in systems with coagulation and fragmentation are obtained from (1.1) when $\partial c(g,t)/\partial t=0$. The integral relations (1.7) represent the additional condition that must be satisfied by the steady-state spectra $c^s(g)$.

2. Confining ourselves to the representation of the operator Q in the form (1.5), we obtain the steady-state spectra in the system with model coagulation and fragmentation

$$K(g,n)=\alpha(gn)^\lambda,\ f(g,n)=\beta(g+n)^\lambda;\ \lambda<2 \tag{2.1}$$

In this case the lifetime $\tau(g)$ of a particle of mass g decreases as g increases, and the spectrum $\theta(n,g)$ of particles into which they are fragmented is constant in the interval $0\leqslant n\leqslant g$:

$$\tau(g)=\frac{2}{\beta g^{\lambda+1}};\quad \theta(n,g)=\frac{2}{g},\quad n\leqslant g \tag{2.2}$$

Introducing the function $v(g) = g^{\lambda}c^s(g)$, we obtain from (1.1) when $\partial c/\partial t = 0$ the equation

$$\alpha\left[\int_0^g v(g-n)\,v(n)\,dn - 2v(g)\int_0^\infty v(n)\,dn\right] = \beta\left[gv(g) - 2\int_g^\infty v(n)\,dn\right]$$

whose solution is $v(g) = \beta\alpha^{-1}e^{-\eta g}$. The value of the constant η is determined from condition (1.7) that is satisfied when $\lambda < 2$. The final expressions for the steady-state spectra $c^s(g)$ has the form

$$c^s(g) = \frac{\beta e^{-\eta g}}{\alpha g^{\lambda}};\quad \eta = \left[\frac{\beta\Gamma(2-\lambda)}{\alpha\rho}\right]^{1/(2-\lambda)},\quad \lambda < 2 \tag{2.3}$$

Note that when $\lambda \geqslant 1$ the denumerable particle concentration

$$N^s = \int_0^\infty c^s(g)\,dg \tag{2.4}$$

formally become infinite. The solution in the special case $\lambda = 0$ similar to (2.3) was obtained in /5/ (where it is presented with some errors). The solution of the discrete version of the kinetic equation similar to (2.3) can be obtained using the method described in /4/.

3. Let us analyse the steady-state spectrum of a system with constant coagulation coefficients $K(g, n) = \alpha$ and with multiple fragmentation defined by the function $\gamma(n, g) = \beta/n$. Constant coagulation coefficients nearly approximate the process of Brownian coagulation /7/, while the function $\gamma = \beta/n$ defines the fragmentation of particles whose lifetime is independent of their dimensions, and the spectral function $\theta(n, g)$ that ensures primarily the splitting of small particles, that is most convenient from the energy point of view /2/.

$$\tau(g) = 1/\beta;\ \theta(n, g) = 1/n,\ n \leqslant g \tag{3.1}$$

Eq.(1.1) for steady-state spectra in this case has the form

$$\alpha\left[\frac{1}{2}\int_0^g c^s(g-n)\,c^s(n)\,dn - c^s(g)\int_0^\infty c^s(n)\,dn\right] + \beta\left[\frac{1}{g}\int_g^\infty c^s(n)\,dn - c^s(g)\right] = 0 \tag{3.2}$$

For sinugular dimension distributions of particles similar to (2.3) when $\lambda \geqslant 1$, the integrals in the coagulation operator S may be divergent, and in the formula for $S(c^s; g)$ they must be taken as the limit

$$S(c^s; g) = \lim_{\varepsilon\to 0}\left[\frac{1}{2}\int_\varepsilon^{g-\varepsilon} c^s(g-n)\,c^s(n)\,dn - c^s(g)\int_\varepsilon^\infty c^s(n)\,dn\right] \tag{3.3}$$

To avoid problems of divergence we use the function $gS(c^s; g)$ in the following form /8/:

$$gS(c^s; g) = \alpha\rho c^s(g) + \lim_{\varepsilon\to 0}\alpha\frac{d^2}{dg^2}\left[\int_\varepsilon^{g-\varepsilon}\varphi_1(g-n)\,\varphi_0(n)\,dn\right] \tag{3.4}$$

$$\varphi_0(g) = \int_g^\infty c^s(n)\,dn,\quad \varphi_1(g) = \int_g^\infty nc^s(n)\,dn$$

Multiplying (3.2) by g and integrating from g to ∞ taking (3.4) into account, we obtain an equation in which it is possible to set $\varepsilon = 0$

$$-\alpha\frac{d}{dg}\left[\int_0^g \varphi_1(g-n)\,\varphi_0(n)\,dn\right] + \alpha\rho\varphi_0(g) - \beta g\varphi_0(g) = 0 \tag{3.5}$$

We use the Laplace transformations to solve (3.5)

$$-\alpha sL(\varphi_0, s)\,L(\varphi_1, s) + \alpha\rho L(\varphi_0, s) + \beta\frac{dL(\varphi_0, s)}{ds} = 0$$

The connection between $L(\varphi_0)$ and $L(\varphi_1)$ is obtained from the relation $gd\varphi_0/dg = d\varphi_1/dg$ which leads to the closed equation for $u(s) = L(\varphi_0, s)$

$$[\beta + \alpha su(s)]\frac{du(s)}{ds} + \alpha u^2(s) = 0;\quad u(0) = \rho \tag{3.6}$$

The solution of (3.6) is the function $u(s)$ defined by the transcendental equation

$$u(s)\exp[\alpha\beta^{-1}su(s)] = \rho \tag{3.7}$$

Using the Burman-Lagrange formula /9/, we obtain from (3.7) the expansion of $u(s)$ in powers of s

$$u(s) = \sum_{m=1}^\infty\left(-\frac{\alpha}{\beta}sm\right)^{m-1}\frac{\rho^m}{m!}$$

The coefficients of this series are expressed in terms of the moments of the distribution function $c^s(g)$

$$u(s) = \sum_{m=1}^{\infty} \frac{(-s)^{m-1}}{m!} M_m; \quad M \equiv \int_0^{\infty} g^m c^s(g)\, dg$$

consequently

$$M_m = (\alpha\beta^{-1}m)^{m-1}\rho^m, \quad m = 1, 2, \ldots \tag{3.8}$$

To find the asymptotic form of $c^s(g)$ when $g \gg 1$, we note that the singular point s_0 of the function $u(s)$ is determined in accordance with (3.6) from the equation $s_0 u(s_0) = -\beta\alpha^{-1}$, the function $u(s_0)$ itself is finite but has an infinite derivative. Using the solution (3.7), we obtain

$$s_0 = -\beta/(\alpha\rho e); \quad u(s_0) = \rho e$$

where e is the base of natural logarithms. To find the type of singularity we neglect in (3.6) the change in s in the proximity of s_0 in comparison with the variation of $u(s)$

$$[\beta + \alpha s_0 u(s)] \frac{du}{ds} + \alpha u^2(s) = 0; \quad \ln \frac{u(s)}{\rho e} + \frac{\rho e}{u(s)} = 2 + \frac{\alpha}{\beta} \rho e s$$

Expanding in series in powers of $\varepsilon(s) = 1 - u(s)/(\rho e)$ and confining ourselves to leading terms, we obtain

$$u(s) \underset{s=s_0}{\approx} \rho e [1 - 2^{1/2} (1 + \alpha\beta^{-1}\rho e s)^{1/2}]$$

This behaviour of the representation $L(\varphi_0, s)$ of the singular point nearest to zero corresponds to the following asymptotic form of $c^s(g)$ when $g \gg 1$ /9/:

$$c^s(g) = -\frac{d\varphi_0(g)}{dg} \underset{g\gg 1}{\approx} \left(\frac{\beta\rho e}{2\pi\alpha}\right)^{1/2} \exp\left(-\frac{g\beta}{\alpha\rho e}\right) g^{-3/2} \tag{3.9}$$

Note that the asymptotic formula (3.9) is in good agreement with formula (3.8) for moments

$$M_m^{as} = (\rho e)^m \left(\frac{\alpha}{\beta}\right)^{m-1} \frac{\Gamma(m - 1/2)}{\sqrt{2\pi}} \underset{m\gg 1}{\approx} \left(\frac{\alpha}{\beta} m\right)^{m-1} \rho^m$$

since the moments M_m with large m must be determined by the "tail" of the spectrum of $c^s(g)$.

To estimate the scale g_{as}, at which the asymptotic form (3.9) begins to act we use the property that for large m the quantities M_m^{as} are basically determined by the spectrum (3.9) at the saddle point $g_m = (m - 3/2)\alpha\beta^{-1}\rho e$. The relative difference between M_m^{as} and M_m does not exceed 10% when $m \geqslant 6$; it is, consequently, possible to assume that $g_{as} = 9/2\,\alpha\beta^{-1}\rho e$.

The behaviour of the steady-state spectrum $c^s(g)$ as $g \to 0$ is determined using (3.5) which we represent in the form

$$\alpha \int_0^g (g - n) c^s (g - n) \varphi_0(n)\, dn = \beta g \varphi_0(g) \tag{3.10}$$

By substituting $c^s(g) \underset{g\to 0}{\approx} Ag^{-p}$ we see that the values of $p < 1$ that ensure the finiteness of the quantity

$$\varphi_0(0) = \int_0^{\infty} c^s(g)\, dg$$

cannot satisfy (3.10) as $g \to 0$. The singular $c^s(g)$ with $p > 1$ also do not agree with (3.10), i.e. the unique non-contradictory value is $p = 1$. Substituting $c^s(g) = A/g$, $\varphi_0(g) = -A \ln g$ into (3.10) and equating the coefficients of leading terms in the region of small particle masses we obtain

$$c^s(g) \underset{g\to 0}{\approx} \beta(\alpha g) \tag{3.11}$$

The integral of the denumerable concentration (2.4) is logarithmically divergent at the lower limit. It is interesting that as $g \to 0$ a steady-state spectrum is formed in the model considered, which is independent of the particle mass per unit volume ρ, and is determined only by the ratio of the rates of the processes of fragmentation and coagulation.

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A DYNAMIC CONTACT PROBLEM FOR AN ELASTIC HALF-PLANE IN THE CASE OF HIGH FREQUENCY OSCILLATIONS*

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Harmonic high frequency oscillations of a rigid stamp coupled without friction to an elastic half-plane are considered. The main difficulty in constructing the high-frequency asymptotic forms is that of carrying out the effective factorization of the kernel of the basic integral equation. A function is proposed, which takes into account all properties of the kernel, enables it to be uniformly approximated and is easily factorized. Such a solution of the problem of approximate factorization makes it possible to write, in a simple explicit form, the principal term of the asymptotic expression of the solution. The nature of the distribution of contact stresses under the stamp is studied, as well as the compliance of the foundation and phase shift between the applied force and the displacement of the stamp.

The problem was studied earlier in /1–4/ for the low-frequency case. Three classes of solutions were constructed in /5/, the low frequency solution, one effective at medium frequencies, and a high-frequency solution. The high frequency solution of /5/ however does not capture the true root-type singularities of the contact stress near the sharp edges of the stamp.

1. As we know /4, 5/, the problem in question can be reduced to the following integral equation:

$$\int_{-1}^{1} \varphi(t) K\left(\frac{x-t}{\lambda}\right) dt = \frac{G}{a} W, \quad |x| \leqslant 1 \tag{1.1}$$

$$K(x) = \frac{1}{2\pi} \int_{\sigma}^{\infty} L_*(u) e^{-iux} du, \quad \lambda^2 = \frac{G}{\rho \varkappa^2 a^2}, \quad \beta^2 = \frac{1-2\nu}{2(1-\nu)}$$

$$L_*(u) = \frac{\sqrt{u^2-\beta^2}}{4u^2\sqrt{u^2-\beta^2}\sqrt{u^2-1} - (2u^2-1)^2}$$

The dependence of all quantities on time is assumed to be of the type $f(x, t) = \mathrm{Re}\,[f(x) e^{-i\varkappa t}]$. In (1.1) $\varphi(x)$ is the contact stress amplitude, W is the stamp oscillation amplitude, λ is a parameter which is small at high frequencies, G, ν are the elastic constants, a is the stamp half-width and $\varkappa$ is the frequency of the oscillations. The initial Eq. (1.1) is equivalent to the following two Eqs. /6/:

$$\int_{0}^{\infty} \omega(t) K(x-t)\, dt = \frac{1}{\lambda} + \int_{0}^{\infty} \left[\omega\left(\frac{2}{\lambda} + \tau\right) - r(\tau)\right] K(x+\tau)\, d\tau \tag{1.2}$$

$$\int_{-\infty}^{\infty} r(t) K\left(\frac{x-t}{\lambda}\right) dt = 1 \tag{1.3}$$

provided that

$$\varphi(x) = \frac{G}{a} W \left[\omega\left(\frac{1-x}{\lambda}\right) - \omega\left(\frac{1-x}{\lambda}\right) - r(x)\right] \tag{1.4}$$

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